A note on capturing curvatures of surfaces by contours (Local and global study of singularity theory of differentiable maps)

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A note on capturing curvatures of surfaces by contours

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Abstract

This is an announcement of the forthcoming paper "Capturing curvatures of surfaces by contours" by the same authors. Given a surface in the Euclidean three space, we give formula for its second and third order information of the surface from curvatures of the three and four contours. The similar formula for space curves are given.

1 Introduction

The Gaussian curvature of a surface is required by the informations of 2-jet of the surface. In [2, 3], Koenderink showed that one can obtain the Gaussian curvature of a surface as the product of the curvature of the contour and the normal curvature along a given direction. This fact suggests that we can obtain some informations of a surface from curvatures of contours of the surface.

Let $s \in \boldsymbol{R}^3$ be a point and let $o_1, o_2 \in \boldsymbol{R}^3$ be two other points. Assume that s is unknown and o_1, o_2 are known. Then one can obtain the coordinate of s by the angles between $\overrightarrow{o_1 s}$, $\overrightarrow{o_1 o_2}$ and between $\overrightarrow{o_2 s}$, $\overrightarrow{o_2 o_1}$. Then it is natural to ask that for a given unknown surface $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ whether we can know the information from the curvatures of contours of the orthogonal projections of f. Without loss of generality, we may assume that f is given by

$$f(u,v) = (u, v, h(u,v)), \quad h(u,v) = \frac{a_{20}}{2}u^2 + \frac{a_{02}}{2}v^2 + \sum_{i+j=3}^{k} \frac{a_{ij}}{i!j!} u^i v^j + O(k+1), \tag{1.1}$$

where $a_{ij} \in \boldsymbol{R}$ $(i,j = 0,1,2,\ldots)$, and $O(k+1)$ stands for the terms whose degrees are greater than k. We call a_{20}, a_{02} (respectively, $a_{30}, a_{21}, a_{12}, a_{03}$) the *second order* (respectively, the *third order*) informations of f at 0. In this paper, we show that we can obtain second order informations of f from the curvatures of contours of three distinct projections, and can obtain third order informations from the curvatures of contours of four distinct projections. More precisely, let us consider a unit vector $\xi \in \boldsymbol{R}^3$ and the projection

$$\pi_\xi(x) = x - \langle x, \xi \rangle \, \xi : \boldsymbol{R}^3 \to \xi^\perp$$

We set $f_\xi = \pi_\xi(f)$. We call the set $S(f_\xi)$ of singular points the *contour generator*, and $f_\xi(S(f_\xi))$ the *contour*. We give formula for a_{20}, a_{02} written by the curvatures of the contours of three distinct directions, and formula for $a_{30}, a_{21}, a_{12}, a_{03}$ written by the curvatures of the contours of four distinct directions.

More primitively, the similar things about space curves will be discussed.

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Throughout the paper, to represent the coefficients of a function, we use the following notation. For a function $h : (\boldsymbol{R}, 0) \to \boldsymbol{R}$, we set

$$(\mathrm{coef}_0(h,t,k) =)\, \mathrm{coef}(h,t,k) = \left(h(0), h'(0), \frac{h''(0)}{2}, \dots, \frac{h^{(k)}(0)}{k!}\right) \quad \left({}' = \frac{d}{dt}\right),$$

namely, if $h = a_0 + \sum_{i=1}^{k}(a_i/i!)t^i$, then $\mathrm{coef}(h,t,k) = (a_0, a_1, \dots, a_k)$.

2 Space curves

Let $\gamma : (\boldsymbol{R}, 0) \to (\boldsymbol{R}^3, 0)$ be a C^∞ curve, and let $\gamma_\xi = \pi_\xi(\gamma)$ for ξ and π given in Introduction. We assume that the curvature of γ does not vanish at 0. Since we are looking for a singular case, we consider the following two cases. The first case is the projection curve γ_ξ has an inflection point, namely, the vector ξ lies in the osculating plane. The second case is one of the projection curve γ_ξ has a singular point, namely, the vector ξ is tangent to γ at 0.

2.1 Projections in the osculating plane

In this section, we consider the case that ξ lies in the osculating plane γ at 0. We give an orientation of γ. Then without loss of generality, we may assume that

$$\gamma(t) = \left(t, \sum_{i=2}^{5} \frac{a_i}{i!}t^i, \sum_{i=3}^{5} \frac{b_i}{i!}t^i\right) + (O(6), O(6), O(6)), \tag{2.1}$$

where $a_i, b_i \in \boldsymbol{R}$ $(i = 2, \dots, 5)$, and $\xi(\theta_1) = (\cos\theta_1, \sin\theta_1, 0)$, where $0 < \theta_1 < \pi$. We give the orientation of $\xi^\perp$ as follows: We take a basis $\{X, Y\}$ of $\xi^\perp$. We say that $\{X, Y\}$ is a positive basis if $\{X, Y, \xi\}$ is a positive basis of $\boldsymbol{R}^3$. We set the orientation of $\pi_{\xi(\theta_1)}(\gamma)$ agreeing that of γ. We set $\pi_{\xi(\theta_1)}(\gamma) = \gamma_{\theta_1}$, s the arc-length of γ_{θ_1}, and set κ_{θ_1} the curvature of $\gamma_{\theta_1} \subset \xi^\perp$ as in a curve in the positively oriented plane $\xi^\perp$. Then by a direct calculation, we have

$$\begin{aligned}\mathrm{coef}(\kappa_{\theta_1}, s, 3) = \Bigg(0, -\frac{b_3}{\sin^3\theta_1}, -\frac{b_4\sin\theta_1 + 6a_2b_3\cos\theta_1}{2\sin^5\theta_1}, \\ -\frac{45a_2^2b_3\cos\theta_1^2 + b_5\sin^2\theta_1 + 10(a_3b_3 + a_2b_4)\sin\theta_1\cos\theta_1}{6\sin^7\theta_1}\Bigg).\end{aligned} \tag{2.2}$$

We take another direction $\theta_2 = \theta_1 + \varphi$ $(0 < \theta_2 < \pi)$, then we may consider $\kappa_{\theta_1}, \kappa_{\theta_2}, \varphi$ are known. We assume that $(d\kappa_{\theta_1}/ds(0), d\kappa_{\theta_2}/ds(0)) \neq (0,0)$. Without loss of generality, we assume $d\kappa_{\theta_2}/ds(0) \neq 0$. Since

$$\frac{d\kappa_{\theta_1}/ds(0)}{d\kappa_{\theta_2}/ds(0)} = \frac{\sin^3(\theta_1 + \varphi)}{\sin^3\theta_1}$$

can be solved as

$$\theta_1 = \cot^{-1}\left(\frac{\left(\dfrac{d\kappa_{\theta_1}/ds(0)}{d\kappa_{\theta_2}/ds(0)}\right)^{1/3} - \cos\varphi}{\sin\varphi}\right) \in (0,\pi),$$

we obtain θ_1 and θ_2. Furthermore, by (2.2), it holds that

$$\sin\theta_i = -\frac{\tilde{b}}{\tilde{\kappa}_{\theta_i}}, \quad (i=1,2) \tag{2.3}$$

where $\tilde{b} = b_3^{1/3}$ and $\tilde{\kappa}_{\theta_i} = (d\kappa_{\theta_i}/ds(0))^{1/3}$. Substituting (2.3) into a trigonometric identity

$$\cos^2(\theta_1-\theta_2) + \sin^2\theta_1 + \sin^2\theta_2 - 2\sin\theta_1\sin\theta_2\cos(\theta_1-\theta_2) - 1 = 0,$$

we get

$$(\tilde{\kappa}_{\theta_1}^2 - 2\cos\varphi\,\tilde{\kappa}_{\theta_1}\tilde{\kappa}_{\theta_2} + \tilde{\kappa}_{\theta_2}^2)\tilde{b}^2 - \sin^2\varphi\,\tilde{\kappa}_{\theta_1}^2\tilde{\kappa}_{\theta_2}^2 = 0. \tag{2.4}$$

Since $\tilde{\kappa}_{\theta_1}^2 - 2\cos\varphi\,\tilde{\kappa}_{\theta_1}\tilde{\kappa}_{\theta_2} + \tilde{\kappa}_{\theta_2}^2 = 0$ if and only if $\varphi = 0, \tilde{\kappa}_{\theta_1} = \tilde{\kappa}_{\theta_2}$ or $\tilde{\kappa}_{\theta_1} = \tilde{\kappa}_{\theta_2} = 0$, and $\tilde{\kappa}_{\theta_1}$, $\tilde{\kappa}_{\theta_2}$, φ are known, (2.4) implies that we obtain b_3. Since

$$\frac{d^2\kappa_{\theta_i}}{ds^2}(0) = -\frac{b_4\sin\theta_i + 6a_2b_3\cos\theta_i}{2\sin^5\theta_i} \quad (i=1,2)$$

is a linear system for a_2, b_4, and $\theta_1 \neq \theta_2$, if $b_3 \neq 0$, we obtain a_2 and b_4 by (2.2). By the similar method, if $b_3 \neq 0$, then we obtain a_3, b_5. In particular, we obtain the information of γ up to 3-order by two projections in the osculation plane.

2.2 Projections by tangential direction with another direction

In this section, we consider the case that ξ is tangent to γ at 0. In this case, $\pi_\xi(\gamma)$ has a singular point at 0. To consider differential geometric invariants of the singular curve, we state the cuspidal curvature of singular plane curves introduced in [5] (see also [6]). Let $c : (\boldsymbol{R}, 0) \to (\boldsymbol{R}^2, 0)$ be a plane curve, and $c'(0) = 0$. The curve c is called to be *A-type* if $c''(0) \neq 0$. Let c be a A-type germ. Then

$$\mu = \frac{\det(c''(0), c'''(0))}{|c''(0)|^{5/2}}.$$

does not depend on the choice of the parameter, and called the *cuspidal curvature.*

Let $\gamma : (\boldsymbol{R}, 0) \to (\boldsymbol{R}^3, 0)$ be a C^∞ curve with the non-zero curvature at 0. We assume that $\pi_\xi(\gamma)$ has a singular point at 0. Since the curvature of γ does not vanish, by the Bouquet theorem, $\pi_\xi(\gamma)$ is the A-type germ at 0. We also assume that there exists an integer N such that $\det(\pi_\xi(\gamma)'', \pi_\xi(\gamma)^{(2N+1)})(0) \neq 0$. We give the positively oriented xyz-coordinate system for $\boldsymbol{R}^3$, and yz-coordinate system for $\xi^\perp$ as follows: We set the y-axis as the direction of $\pi_\xi(\gamma)''(0)$, and set the x-axis as the direction of ξ. We give an orientation $\pi_\xi(\gamma)$ so that $\det(\pi_\xi(\gamma)'', \pi_\xi(\gamma)^{(2N+1)})(0) > 0$, and also that of γ agreeing with that of

$\pi_\xi(\gamma)$. Then we may assume that γ is given by (2.1) with $a_2 > 0, b_3 \geq 0$, and we have $\mu = b_3/a_2^{3/2}$. On the other hand, we consider a unit vector $\xi = (\sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2, \cos\theta_1)$. Since we take the above xyz-coordinate, θ_1, θ_2 is known. Then the curvature κ_ξ of the plane curve $\pi_\xi(\gamma)$ satisfies

$$\begin{aligned}\mathrm{coef}(\kappa_\xi, s, 1) = \Bigg(& \frac{a_2\cos\theta_1}{(\cos^2\theta_1\cos^2\theta_2 + \sin^2\theta_2)^{3/2}}, \\ & \frac{1}{(\cos^2\theta_1\cos^2\theta_2 + \sin^2\theta_2)^3}\Big(- b_3\cos^2\theta_1\cos^2\theta_2\sin\theta_1\sin\theta_2 \\ & \quad - b_3\sin\theta_1\sin^3\theta_2 + \cos^3\theta_1\cos\theta_2(a_3\cos\theta_2 - 3a_2^2\sin\theta_2) \\ & \quad + \cos\theta_1\sin\theta_2(3a_2^2\cos\theta_2 + a_3\sin\theta_2)\Big)\Bigg).\end{aligned} \tag{2.5}$$

Since we know $\mu = b_3/a_2^{3/2}$ and θ_1, θ_2, if $\cos\theta_1 \neq 0$, then we obtain a_2 and b_3 by the first component of (2.5). Furthermore, we also obtain a_3 by the second component of (2.5) under the assumption $\cos\theta_1 \neq 0$.

3 Surfaces

Let $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ be a C^∞ surface, and ξ a unit vector which is tangent to f at 0. Then without loss of generality, we may assume f is written in the form (1.1) with $a_{20}a_{02} \neq 0$, $a_{20} > 0$, and assume $\xi(\theta_1) = (\cos\theta_1, \sin\theta_1, 0)$, where $0 < \theta_1 < \pi$. We set the unit normal vector ν of f satisfies $\nu(0,0) = (1,0,0)$. Then the set of singular points S of the map $\pi_{\xi(\theta_1)} \circ f$ is

$$\{(u,v) \mid \cos\theta_1 h_u + \sin\theta_1 h_v = 0\}. \tag{3.1}$$

We assume that $p(\theta_1) \neq 0$ where

$$p(\theta_1) = a_{20}\cos^2\theta_1 + a_{02}\sin^2\theta_1. \tag{3.2}$$

This assumption implies that the direction $\xi(\theta_1)$ is not the asymptotic direction of f. By this assumption,

$$\Big((\cos\theta_1 h_u + \sin\theta_1 h_v)_u, (\cos\theta_1 h_u + \sin\theta_1 h_v)_v\Big)(0,0) = (a_{20}\cos\theta_1, a_{02}\sin\theta_1) \neq (0,0),$$

there exists a regular parametrization of S. For the sake of taking this parametrization, we set an orientation of S as follows. First, we give an orientation of the normal plane $\xi(\theta_1)^\perp$ of $\xi(\theta_1)$ as

$$X = (-\sin\theta_1, \cos\theta_1, 0), \quad Y = (0,0,1)$$

is a positive basis. Next, put an orientation of $(\pi_{\xi(\theta_1)} \circ f)(S)$ as it is agreeing to the direction of X (Figure 1), and also put that of S agreeing to $(\pi_{\xi(\theta_1)} \circ f)(S)$. Since $a_{02}\sin\theta_1 \neq 0$, we can take a parametrization $C(t) = (t, c(t))$. Then

$$(\pi_{\xi(\theta_1)}(f) \circ C)(t) = {}^t\begin{pmatrix} t\sin^2\theta_1 - c(t)\cos\theta_1\sin\theta_1 \\ c(t) - t\cos\theta_1\sin\theta_1 - c(t)\sin^2\theta_1 \\ h(t, c(t)) \end{pmatrix},$$

and

$$(\pi_{\xi(\theta_1)}(f) \circ C)'(0) = {}^t\begin{pmatrix} \sin^2\theta_1 - c'(t)\cos\theta_1\sin\theta_1 \\ -\cos\theta_1\sin\theta_1 - c(t)'\cos^2\theta_1 \\ h(t, c(t))' \end{pmatrix}(0)$$
$$= {}^t\begin{pmatrix} \sin^2\theta_1 - c'(0)\cos\theta_1\sin\theta_1 \\ -\cos\theta_1\sin\theta_1 - c(0)'\cos^2\theta_1 \\ 0 \end{pmatrix} = (-\sin\theta_1 + c'(0)\cos\theta_1)X.$$

By (3.1), it holds that

$$\mathrm{coef}(c(t), 2, t) = \Bigg(0, -\frac{a_{20}\cos\theta_1}{a_{02}\sin\theta_1}, \frac{1}{a_{02}^3\sin^3\theta_1}\Big(- a_{12}a_{20}^2\cos^3\theta_1 - a_{03}a_{20}^2\cos^2\theta_1\sin\theta_1$$
$$+ 2a_{02}a_{20}a_{21}\cos^2\theta_1\sin\theta_1 + 2a_{02}a_{12}a_{20}\cos\theta_1\sin^2\theta_1$$
$$- a_{02}^2a_{30}\cos\theta_1\sin^2\theta_1 - a_{02}^2a_{21}\sin^3\theta_1\Big)\Bigg).$$

Then we see that

$$(\pi_{\xi(\theta_1)} \circ f \circ C)'(0) = \frac{-1}{a_{02}\sin\theta_1}p(\theta_1). \tag{3.3}$$

Let s be the arc-length parameter of $\pi_{\xi(\theta_1)}(S)$ where the orientation is given by the above manner. Thus we remark that by (3.3), if $a_{02}\sin\theta_1 p(\theta_1)$ is negative, s is the opposite direction with the above parameter t. The curvature k_{θ_1} of the contour satisfies

$$\mathrm{coef}(k_{\theta_1}, 1, s) = \left(\frac{a_{20}a_{02}}{p(\theta_1)}, \frac{q(\theta_1)}{p(\theta_1)^3}\right), \tag{3.4}$$

where

$$q(\theta_1) = a_{03}a_{20}^3\cos^3\theta_1 - 3a_{02}a_{12}a_{20}^2\cos^2\theta_1\sin\theta_1 + 3a_{02}^2a_{20}a_{21}\cos\theta_1\sin^2\theta_1 - a_{02}^3a_{30}\sin^3\theta_1. \tag{3.5}$$

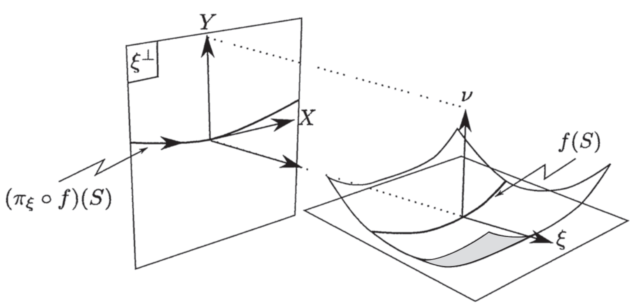

Figure 1: Orientations of $\xi^\perp$ and contour.

Remark that if $a_{20}a_{02} \neq 0$ and $p(\theta_1) \neq 0$ then $q(\theta_1) = 0$ if and only if the contour has a vertex at $(\pi_{\xi(\theta_1)} \circ f \circ C)(0)$, and that $\xi(\theta_1) = (\cos\theta_1, \sin\theta_1, 0)$ is called the cylindrical direction of f at the origin (see [1] for details).

3.1 Obtaining third order jet of surfaces

Let us consider how many directions we need to obtain third order jet of surfaces. We take another direction θ_2 which satisfies $p(\theta_2) \neq 0$. By (3.4) we get

$$\cos 2\theta_i = \frac{-2a_{20}a_{02} + (a_{20} + a_{02})k_{\theta_i}}{(a_{02} - a_{20})k_{\theta_i}} \quad (i = 1, 2).$$

Substituting these formulas into a trigonometric identity

$$\cos^2 2(\theta_i - \theta_j) + \cos^2 2\theta_i + \cos^2 2\theta_j - 2\cos 2(\theta_i - \theta_j)\cos 2\theta_i \cos 2\theta_j - 1 = 0,$$

we get $P_{ij}(G, M) = 0$ where

$$\begin{aligned} P_{ij}(G, M) := & \quad \left(M_{ij}^2 - G_{ij}\cos^2(\theta_i - \theta_j)\right)G^2 - 2G_{ij}M_{ij}\sin^2(\theta_i - \theta_j)GM \\ & + G_{ij}^2 \sin^4(\theta_i - \theta_j)M^2 + G_{ij}^2\cos^2(\theta_i - \theta_j)\sin^2(\theta_i - \theta_j)G \\ = & \quad (G, M)\, Q_{ij}\, {}^t(G, M) + G_{ij}^2\cos^2(\theta_i - \theta_j)\sin^2(\theta_i - \theta_j)G \end{aligned}$$

and

$$M = \frac{a_{20} + a_{02}}{2}, \quad G = a_{20}a_{02}, \quad M_{ij} = \frac{k_{\theta_i} + k_{\theta_j}}{2}, \quad G_{ij} = k_{\theta_i}k_{\theta_j}, \tag{3.6}$$

$$Q_{ij} = \begin{pmatrix} M_{ij}^2 - G_{ij}\cos^2(\theta_i - \theta_j) & -G_{ij}M_{ij}\sin^2(\theta_i - \theta_j) \\ -G_{ij}M_{ij}\sin^2(\theta_i - \theta_j) & G_{ij}^2\sin^4(\theta_i - \theta_j) \end{pmatrix}.$$

Since $P_{ij}(G, M) = 0$ is a quadratic curve, generally the values of G and M should be determined by the curvatures of the apparent contours from distinct three directions. In fact, we get the following formula with respect to $G, M, G_{12}, G_{23}, G_{31}, \theta_1, \theta_2, \theta_3$.

First, a system of equations as below holds:

$$W \begin{pmatrix} G^2 \\ GM \\ M^2 \end{pmatrix} = Gb$$

where $W = (w_1, w_2, w_3)$ with

$$w_1 = \begin{pmatrix} M_{12}^2 - G_{12}\cos^2(\theta_1 - \theta_2) \\ M_{23}^2 - G_{23}\cos^2(\theta_2 - \theta_3) \\ M_{31}^2 - G_{31}\cos^2(\theta_3 - \theta_1) \end{pmatrix}, \tag{3.7}$$

$$w_2 = -\begin{pmatrix} 2G_{12}M_{12}\sin^2(\theta_1 - \theta_2) \\ 2G_{23}M_{23}\sin^2(\theta_2 - \theta_3) \\ 2G_{31}M_{31}\sin^2(\theta_3 - \theta_1) \end{pmatrix}, \tag{3.8}$$

$$w_3 = \begin{pmatrix} G_{12}^2\sin^4(\theta_1 - \theta_2) \\ G_{23}^2\sin^4(\theta_2 - \theta_3) \\ G_{31}^2\sin^4(\theta_3 - \theta_1) \end{pmatrix}, \tag{3.9}$$

and

$$b = \begin{pmatrix} G_{12}^2\cos^2(\theta_1 - \theta_2)\sin^2(\theta_1 - \theta_2) \\ G_{23}^2\cos^2(\theta_2 - \theta_3)\sin^2(\theta_2 - \theta_3) \\ G_{31}^2\cos^2(\theta_3 - \theta_1)\sin^2(\theta_3 - \theta_1) \end{pmatrix}. \tag{3.10}$$

Assume $\theta_i \neq \theta_j$ and $G_{ij} \neq 0$ for $i \neq j$, then the determinant of W is expressed as

$$\det W = -2G_{12}^2 G_{23}^2 G_{31}^2 \sin^2(\theta_1-\theta_2)\sin^2(\theta_2-\theta_3)\sin^2(\theta_3-\theta_1)\det V$$

with

$$V = \begin{pmatrix} \frac{M_{12}^2 - G_{12}\cos^2(\theta_1-\theta_2)}{G_{12}^2\sin^2(\theta_1-\theta_2)} & \frac{M_{12}}{G_{12}} & \sin^2(\theta_1-\theta_2) \\ \frac{M_{23}^2 - G_{23}\cos^2(\theta_2-\theta_3)}{G_{23}^2\sin^2(\theta_2-\theta_3)} & \frac{M_{23}}{G_{23}} & \sin^2(\theta_2-\theta_3) \\ \frac{M_{31}^2 - G_{31}\cos^2(\theta_3-\theta_1)}{G_{31}^2\sin^2(\theta_3-\theta_1)} & \frac{M_{31}}{G_{31}} & \sin^2(\theta_3-\theta_1) \end{pmatrix}.$$

With Cramer's rule, we get

$$G = \frac{\det W_1}{\det W}, \quad M = \frac{\det W_2}{\det W}, \tag{3.11}$$

where $W_1 = (b, w_2, w_3)$, $W_2 = (w_1, b, w_3)$.

Especially, $\det W_1$ is expressed as

$$-2G_{12}^2 G_{23}^2 G_{31}^2 \sin^2(\theta_1-\theta_2)\sin^2(\theta_2-\theta_3)\sin^2(\theta_3-\theta_1)\det L$$

with

$$L = \begin{pmatrix} \cos^2(\theta_1-\theta_2) & \frac{M_{12}}{G_{12}} & \sin^2(\theta_1-\theta_2) \\ \cos^2(\theta_2-\theta_3) & \frac{M_{23}}{G_{23}} & \sin^2(\theta_2-\theta_3) \\ \cos^2(\theta_3-\theta_1) & \frac{M_{31}}{G_{31}} & \sin^2(\theta_3-\theta_1) \end{pmatrix}$$

and the numerator of M is

$$2G_{12}^2 G_{23}^2 G_{31}^2 \sin^2(\theta_1-\theta_2)\sin^2(\theta_2-\theta_3)\sin^2(\theta_3-\theta_1)\det P$$

with

$$P = \begin{pmatrix} \frac{M_{12}^2 - G_{12}\cos^2(\theta_1-\theta_2)}{G_{12}^2\sin^2(\theta_1-\theta_2)} & \cos^2(\theta_1-\theta_2) & \sin^2(\theta_1-\theta_2) \\ \frac{M_{23}^2 - G_{23}\cos^2(\theta_2-\theta_3)}{G_{23}^2\sin^2(\theta_2-\theta_3)} & \cos^2(\theta_2-\theta_3) & \sin^2(\theta_2-\theta_3) \\ \frac{M_{31}^2 - G_{31}\cos^2(\theta_3-\theta_1)}{G_{31}^2\sin^2(\theta_3-\theta_1)} & \cos^2(\theta_3-\theta_1) & \sin^2(\theta_3-\theta_1) \end{pmatrix}.$$

Since we may regard $\theta_1-\theta_2$, $\theta_1-\theta_3$, k_{θ_1}, k_{θ_2}, k_{θ_3} are known, we obtain θ_1, and this implies we obtain θ_2, θ_3. This also implies that we obtain G_{ij} $(ij = 12, 23, 31)$ and w_1, w_2, w_3, b (see (3.6), (3.7), (3.8), (3.9), (3.10)). Furthermore, we obtain G and M by (3.11). Since $G = a_{20}a_{02}$ and $M = (a_{20}+a_{02})/2$, we obtain a_{20} and a_{02}.

Next let us consider the third order terms of the surface. Let us take four distinct directions

$$\theta_1,\ \theta_2,\ \theta_3,\ \theta_4.$$

Then by (3.4) and (3.5), we see that

$$A\begin{pmatrix} a_{30} \\ a_{21} \\ a_{12} \\ a_{03} \end{pmatrix} = v,$$

where $A = (a_1, a_2, a_3, a_4)$ and

$$
\begin{aligned}
a_1 &= -a_{02}^3 \,{}^t\!\left(\sin^3\theta_1,\ \sin^3\theta_2,\ \sin^3\theta_3,\ \sin^3\theta_4 \right),\\
a_2 &= 3a_{20}a_{02}^2 \,{}^t\!\left(\sin^2\theta_1\cos\theta_1,\ \sin^2\theta_2\cos\theta_2,\ \sin^2\theta_3\cos\theta_3,\ \sin^2\theta_4\cos\theta_4 \right),\\
a_3 &= -3a_{20}^2a_{02} \,{}^t\!\left(\sin\theta_1\cos^2\theta_1,\ \sin\theta_2\cos^2\theta_2,\ \sin\theta_3\cos^2\theta_3,\ \sin\theta_4\cos^2\theta_4 \right),\\
a_4 &= a_{20}^3 \,{}^t\!\left(\cos^3\theta_1,\ \cos^3\theta_2,\ \cos^3\theta_3,\ \cos^3\theta_4 \right),\\
v &= {}^t\!\left(p(\theta_1)^3\frac{d\kappa_{\theta_1}}{ds}(0),\ p(\theta_2)^3\frac{d\kappa_{\theta_2}}{ds}(0),\ p(\theta_3)^3\frac{d\kappa_{\theta_3}}{ds}(0),\ p(\theta_4)^3\frac{d\kappa_{\theta_4}}{ds}(0) \right),
\end{aligned}
$$

where ${}^t(\ )$ stands for the matrix transportation. Since $\det A = 9a_{20}^6a_{02}^6\prod_{i<j}\sin(\theta_i - \theta_j)$, and $\theta_1, \ldots, \theta_4$ are distinct, $a_{20}a_{02} \neq 0$, it holds that $\det A \neq 0$. By Cramer's rule, we get

$$
a_{30} = \frac{\det A_1}{\det A},\quad a_{21} = \frac{\det A_2}{\det A},\quad a_{12} = \frac{\det A_3}{\det A},\quad a_{03} = \frac{\det A_4}{\det A},
$$

where $A_1 = (v, a_2, a_3, a_4)$, $A_2 = (a_1, v, a_3, a_4)$, $A_3 = (a_1, a_2, v, a_4)$, $A_4 = (a_1, a_2, a_3, v)$. This implies that we obtain $a_{30}, a_{21}, a_{12}, a_{03}$ by k_{θ_i} $(i = 1, 2, 3, 4)$.

3.2 Obtaining Gaussian curvature

According to Section 3.1, we can obtain all of the the second order information of the surface by the contour of projections from distinct three directions. In particular we can obtain the Gaussian curvature. In this section, we discuss existence of two directions such that the product of the curvatures of the contours along these directions is the Gaussian curvature $K = a_{20}a_{02}$.

By (3.4), we have

$$
k_{\theta_1}k_{\theta_2} = \frac{a_{20}^2a_{02}^2}{(a_{20}\cos^2\theta_1 + a_{02}\sin^2\theta_1)(a_{20}\cos^2\theta_2 + a_{02}\sin^2\theta_2)}.
$$

Hence if

$$
\frac{a_{20}a_{02}}{(a_{20}\cos^2\theta_1 + a_{02}\sin^2\theta_1)(a_{20}\cos^2\theta_2 + a_{02}\sin^2\theta_2)} = 1, \tag{3.12}
$$

then $K = k_{\theta_1}k_{\theta_2}$. If θ_1, θ_2 satisfies (3.12), then we say that $\xi_{\theta_1}, \xi_{\theta_2}$ are *contour-conjugate* each other. Now we consider the existence of the contour-conjugate. Since (3.12) is equivalent to

$$
\left(\frac{\cos\theta_2}{\sin\theta_2}\right)^2 = \frac{a_{02}\sin^2\theta_1}{a_{20}\cos^2\theta_1},
$$

it holds that if $K > 0$ then any direction has two contour-conjugate, and if $K < 0$ there are no contour-conjugate for any direction.

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